Character Table Isomorphisms

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joint work with Steve Goldstein and Michael Stemper

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Some Philosophy

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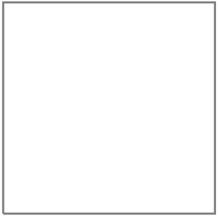
- If we write $\sqrt{9}$ we really mean 3.
- What about $\sqrt{-1}$?

Group

- Groups give us a language to talk about basic properties of "isomorphism" and equivalence.

- Given an object T we can talk about equivalent objects to T.



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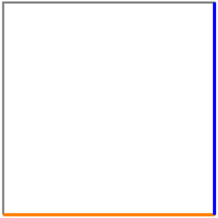

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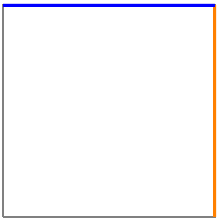

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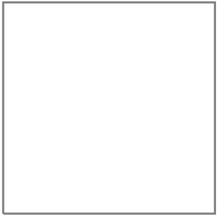

Characters

- Associated to each finite group is an object called a character table.
- The characters are the shadows of the group.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 2 & 0 & \zeta+\zeta^4 & \zeta^2+\zeta^3 \\ 2 & 0 & \zeta^2+\zeta^3 & \zeta+\zeta^4 \end{pmatrix},$$

$\zeta^5 = 1$, i.e., $\zeta = e^{(2\pi i/5)}$.

Shadows

Building a Database

- To help us understand what information about a group G is recoverable from its character table, we are building a database of small finite groups with the same character tables.

- We want to compare about 450,000,000 character tables.

Comparing Two Tables

- The character table of a group G has no canonical ordering, i.e., there is no canonical way of picking which column or row appears where.

- Given two n-by-n character tables M and N. We say $M = N$ if some permutation of the row and columns of M equals the table N.

Comparing Two Tables

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 2 & -1 & 1 & -1 \\ -2 & 1 & -1 & 1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 1 & -1 \\ 2 & 1 & -1 & 1 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ \color{magenta}{2} & \color{magenta}{-1} & \color{magenta}{1} & \color{magenta}{-1} \\ -2 & 1 & -1 & 1 \end{pmatrix} =^{?} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 1 & -1 \\ \color{magenta}{2} & \color{magenta}{1} & \color{magenta}{-1} & \color{magenta}{1} \end{pmatrix}$$

- We encode the table as a graph and run graph isomorphism.

Encoding as a Graph

- Consider the table:

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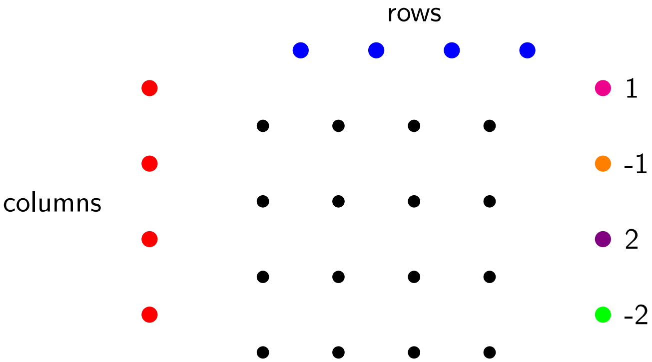

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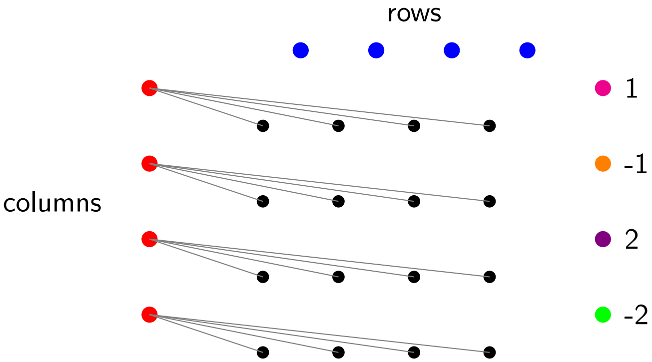

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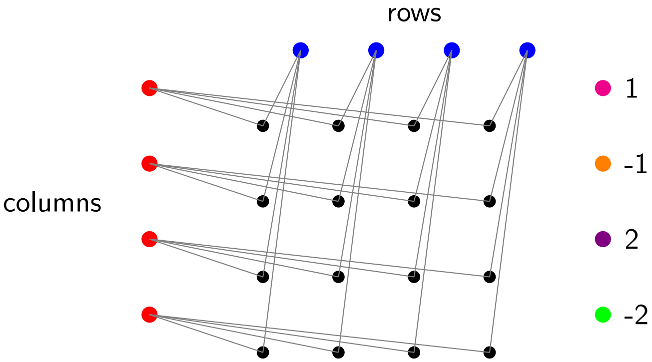

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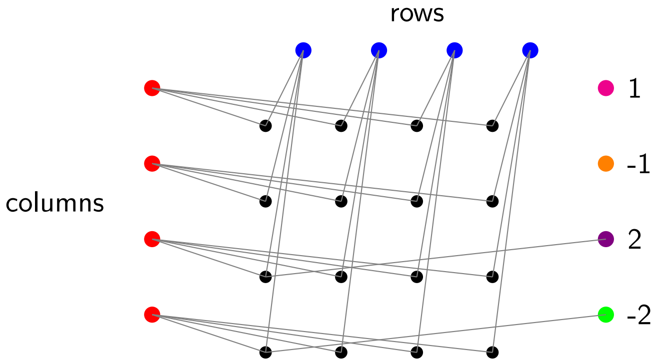

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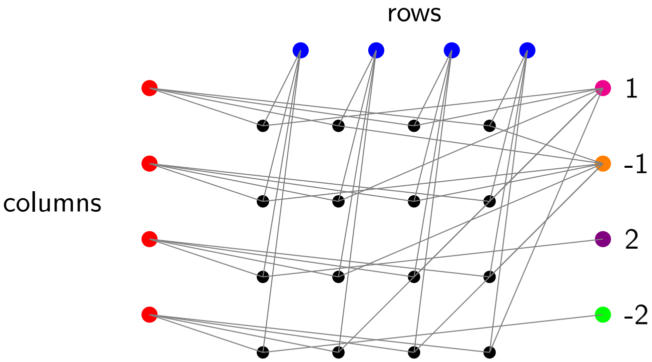

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- The hash is the multiset of rows, where each is a multiset.

$$\{\{-1^2, 1^2\}^2, \{-1^2, 1, 2\}, \{-2, -1, 1^2\}\}.$$

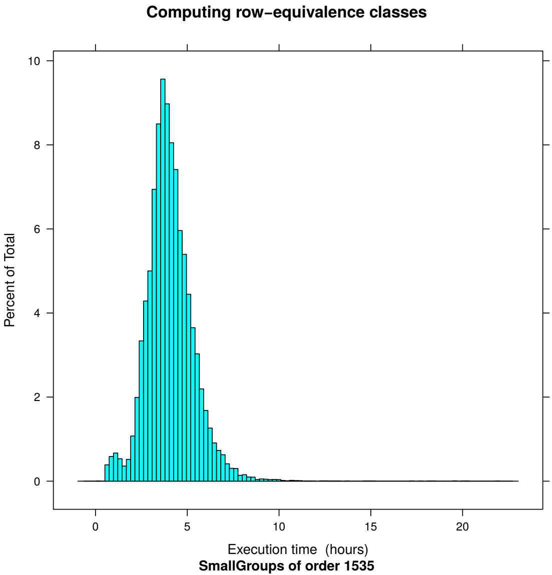
Method

- We run an intial hash:
  - Given a group $\rightarrow$
  - Construct Table $\rightarrow$
  - Create Hash.
- SmallGroup(512,64889569) gives `2dff0c4ba891481cd4fa6e2dc65f298c`.
- SmallGroup(512,64889570) gives `cd246c40463c53d07d13052186170424`.
- SmallGroup(512,54890438) gives `2dff0c4ba891481cd4fa6e2dc65f298c`.

Method

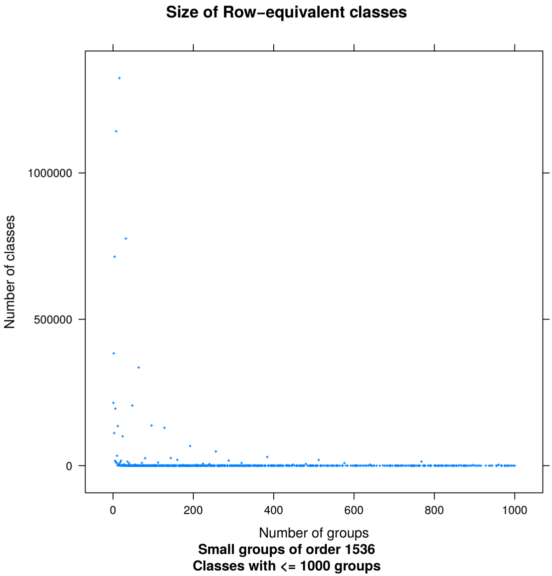
- For each hash bucket run an all against all.
- Each bucket is mostly a single job.

Results

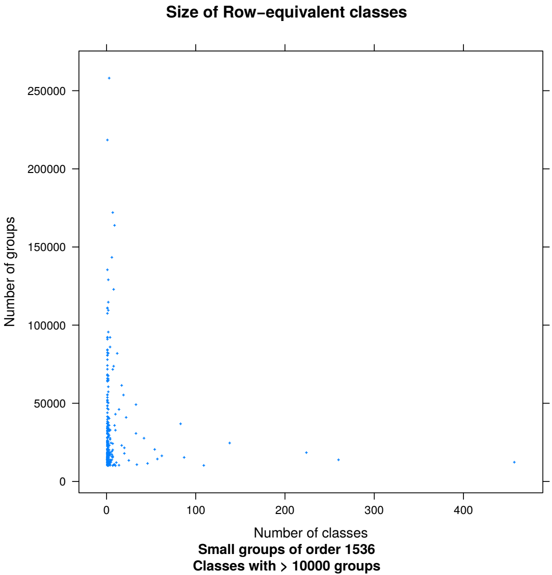


Computing row−equivalence classes
Percent of Total
0
2
4
6
8
10
0
5
10
15
20
Execution time (hours)
SmallGroups of order 1535

Results

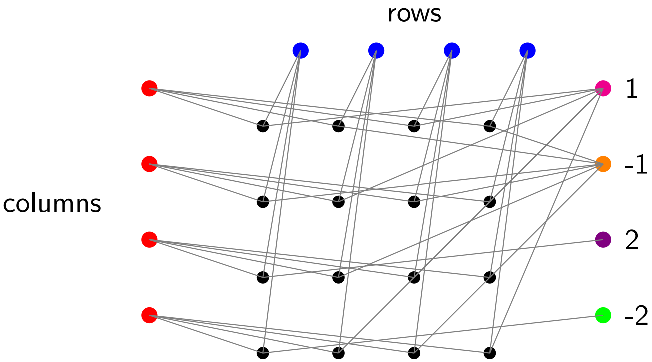

Size of Row−equivalent classes
Number of classes
1000000
500000
0
0
200
400
600
800
1000
Number of groups
Small groups of order 1536
Classes with <= 1000 groups

Results


Size of Row−equivalent classes
250000
200000
150000
100000
50000
0
Number of groups
0
100
200
300
400
Number of classes
Small groups of order 1536
Classes with > 10000 groups

Acknowledgements

- We are grateful to the GAP and HTC-Condor community for project support and troubleshooting.



Acknowledgements

- Thank you for your time.